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Translated by M.D.F.

UDC 539.3

METHOD OF ORTHOGONAL POLYNOMIALS IN PLANE ANTISYMMETRIC MIXED PROBLEMS OF ELASTICITY THEORY WITH TWO CONTACT SECTIONS

PMM Vol. 38, №2, 1974, pp. 331-338

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(Received March 23, 1973)

The possibility is shown of applying the method of orthogonal polynomials to solve some integral equations of a special kind if the eigenfunctions of the integral operator corresponding to the principal (singular) part of the kernel are unknown. Use of the classical scheme [1 – 3] is impossible in this case. However, by using modified Chebyshev polynomials, an integral equation of the form

$$\int_k^1 \varphi(\xi) \ln \left| \frac{\xi + x}{\xi - x} \right| d\xi = \pi f(x) - \int_k^1 \varphi(\xi) G(\xi, x, \lambda)\, d\xi \tag{0.1}$$

$$G(\xi, x, \lambda) = \xi x G_*(\xi, x, \lambda), \quad k \leqslant x \leqslant 1, \quad \lambda \in (0, \infty), \quad k \in (0, 1)$$

is successfully reduced to an infinite algebraic system of the first kind convenient for approximate solution. Here λ, k are dimensionless parameters, G_* is a continuous, even, and symmetric function in ξ, x. Plane antisymmetric mixed problems of elasticity theory with two contact sections, odd in x, reduce to equations of the type (0.1). The odd function $f(x)$ describes the shape of the boundary layer on the contact section $k \leqslant |x| \leqslant 1$ altered under the effect of stamps.

Considered as an illustration is the problem of impressing two flat stamps into a strip.

1. Representing the function $f(x) = f_0(x) + \beta \operatorname{sgn} x$, we seek the solution $\varphi(\xi)$ of (0.1) as

$$\varphi(\xi) = \varphi_0(\xi) + \varphi_1(\xi) \tag{1.1}$$

$$\int_k^1 \varphi_0(\xi) \ln \left| \frac{\xi + x}{\xi - x} \right| d\xi = \pi f_0(x) \qquad (k \leqslant x \leqslant 1) \tag{1.2}$$

Here $\varphi_0(\xi)$, the solution of the integral equation (1.2), is given by formulas in [4] in which it is assumed that $x / a = x$, $\xi / a = \xi$, $b / a = k$, $a = 1$. We have

$$\varphi_0(x) = \frac{2 \operatorname{sgn} x}{\pi g(x)} \left[M_0 - \int_k^1 \frac{g(\xi) f_0'(\xi) \xi}{\xi^2 - x^2}\, d\xi \right] \tag{1.3}$$

$$M_0 = \int_k^1 \varphi_0(x)\, x dx = \int_k^1 \left[\frac{E(k)}{K(k)} - 1 + x^2 \right] \frac{f_0(x)}{g(x)}\, dx$$

$$P_0 = \int_k^1 \varphi_0(x)\,dx = \frac{1}{K(k)} \int_k^1 \frac{f_0(x)}{g(x)}\,dx, \quad g(x) = \sqrt{(1-x^2)(x^2-k^2)}$$

Here $K(k)$, $E(k)$ are the complete elliptic integrals of the first and second kind, respectively. Then the function $\varphi_1(x)$ is found from (0.1) in which

$$\psi(x) = \beta \operatorname{sgn} x - \frac{1}{\pi} \int_k^1 \varphi_0(\xi)\, G(\xi, x, \lambda)\, d\xi \tag{1.4}$$

must be taken in place of $f(x)$. The solution of the integral equation (0.1) has a singularity of type $g^{-1}(x)$, and only one eigenfunction $\Phi_0(x) = \operatorname{sgn} x$ with weight $[\pi g(x)]^{-1}$ and eigennumber $K(k)$ has successfully been sought for the integral operator in the left side of (0.1). Let us seek the solution $\varphi_1(x)$ in the form

$$\varphi_1(x) = \Phi(x)\,[\,|x|\,g(x)]^{-1} \tag{1.5}$$

Let us seek the function $\Phi(x)$, continuous with all derivatives for $x \in [k, 1]$ as a series in modified Chebyshev polynomials of the first kind $T_i^*(x)$ which forms a system with weight $[\pi x g(x)]^{-1}$ orthonormalized in the segment $[k, 1]$:

$$\Phi(x) = \sum_{i=0}^{\infty} a_i T_i^*(x), \qquad T_i^*(x) = x T_{2i}\left(\sqrt{\frac{x^2-k^2}{1-k^2}}\right) \tag{1.6}$$

Let the operator L_-

$$L_-(\varphi) = \frac{1}{\pi} \int_k^1 \varphi(\xi) \ln\left|\frac{\xi+x}{\xi-x}\right| \frac{d\xi}{\xi g(\xi)} \tag{1.7}$$

operate on the function $T_n^*(\xi)$. Integrating, we obtain the equality (C_n, C_{ni} are certain constants)

$$L_-(T_n^*) = C_n \operatorname{sgn} x + \sum_{i=0}^{n-1} C_{ni} T_i^*(x) \tag{1.8}$$

We expand the function $G(\xi, x, \lambda)$ in a double series in the polynomials $T_i^*(x)$

$$G(\xi, x, \lambda) = \sum_{i,j=0}^{\infty} \alpha_{ij} e_{ij} T_i^*(x)\, T_j^*(\xi) \tag{1.9}$$

$$\alpha_{00} = 1, \quad \alpha_{i0} = \alpha_{0j} = 2, \quad \alpha_{ij} = 4$$

Substituting (1.9) into (1.4) and integrating, we obtain

$$\psi(x) = \beta \operatorname{sgn} x + \sum_{i=0}^{\infty} \beta_i b_i T_i^*(x) \quad \left(\beta_0 = \frac{1}{2},\ \beta_i = 1\right) \tag{1.10}$$

By using the property of orthogonality of the polynomials $T_i^*(x)$ and the change of variables

$$x = \sqrt{1 - k'^2 \sin^2 y}, \quad \xi = \sqrt{1 - k'^2 \sin^2 \eta}, \quad k' = \sqrt{1-k^2}$$

we obtain an expression to determine the coefficients b_i and e_{ij} (it is assumed that the coefficient β is known)

$$b_i = \frac{2}{\pi} \int_0^{\pi} \frac{\psi^*(y) \cos 2iy}{\sqrt{1 - k'^2 \sin^2 y}}\, dy \tag{1.11}$$

$$e_{ij} = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \frac{G^*(\eta, y, \lambda)\cos 2iy \cos 2j\eta}{\sqrt{(1-k'^2\sin^2 y)(1-k'^2\sin^2\eta)}}\,dy\,d\eta$$

$$\psi^*(y) = \psi(\sqrt{1-k'^2\sin^2 y}) - \beta$$

$$G^*(\eta, y, \lambda) = G(\sqrt{1-k'^2\sin^2\eta}, \quad \sqrt{1-k'^2\sin^2 y}, \lambda)$$

Using the expansion (1.6), we substitute (1.5), (1.9), (1.10) into the integral equation for $\varphi_1(x)$; after performing all the necessary operations, equating the coefficients of sgn x and the polynomials $T_i^*(x)$ of the same order, we obtain an infinite system of algebraic equations in the coefficients a_i

$$\sum_{j=0}^{\infty} C_j a_j = \beta \tag{1.12}$$

$$\sum_{j=0}^{\infty} e_{ij}a_j + \gamma_i \sum_{j=i+1}^{\infty} C_{ji}a_j = b_i \quad (\gamma_0 = 2,\ \gamma_i = 1,\ i = 0, 1, 2, \ldots)$$

2. The coefficients C_n and C_{ni}, found for each n from (1.8), which is tedious enough, enter into the system (1.12). No formulas expressing C_n and C_{ni} directly in terms of k for arbitrary n have been found.

We derive formulas expressing the interrelation between the coefficients C_n and C_{ni} To this end, let us take the function $f(x)$ for (1.2) as the right side of (1.8) and by using (1.3) we find the solution $\varphi_0(x)$ of (1.2) for this case and we equate the expression $|x|\ g(x)\ \varphi_0(x)$ to the polynomial $T_n^*(x)$. We hence use the relationship (T_{2n} are Chebyshev polynomials in the form of a sum [5])

$$T_{2n}(u) = \sum_{i=0}^{n} \frac{(-1)^i n}{2n-i}\binom{2n-i}{i}(2u)^{2n-2i}$$

$$\int_0^1 \frac{\sqrt{1-v^2}}{u^2-v^2}\,dv = \frac{\pi}{2} \qquad (0 \leqslant u \leqslant 1)$$

Using the change of variable $x = \sqrt{k^2 + k'^2}u$, after manipulation we obtain

$$\begin{aligned} T_{2n}(\sqrt{u}) = {} & k'^2 \sum_{i=0}^{n-1} C_{ni}\Big\{\sum_{j=0}^{i} \frac{(-1)^j i 4^{i-j}}{2i-j}\binom{2i-j}{j}\Big[u^{i-j+1} - \frac{1}{2}u^{i-j} - S_{1j}(u)\Big] - \\ & 4i\sum_{j=1}^{i}(-1)^j 4^{i-j}\binom{2i-j}{j-1}\Big[u^{i-j+2} + \Big(\frac{1}{k'^2} - \frac{3}{2}\Big)u^{i-j+1} - \frac{1}{2}\Big(\frac{k}{k'}\Big)^2 u^{i-j} - \\ & \Big(\frac{k}{k'}\Big)^2 S_{1j}(u) - S_{2j}(u)\Big]\Big\} + \Big[\frac{E(k)}{K(k)} - \frac{k'^2}{2}\Big]C_{n0} + \frac{k'^2}{4}C_{n1} + \frac{C_n}{K(k)} \end{aligned} \tag{2.1}$$

$$S_{1j} = \sum_{m=1}^{i-j}\sum_{s=0}^{m-1}(-1)^s\binom{i-j}{m}\binom{m-1}{s}a_{ms}u^{i-j-m+s}$$

$$S_{2j}=\sum_{m=0}^{i-j}\sum_{s=0}^{m}(-1)^s\binom{i-j+1}{m+1}\binom{m}{s}b_{ms}u^{i-j-m+s}$$

$$a_{ms}=\frac{(2m-2s-1)!!}{(2m-2s+2)!!},\qquad b_{ms}=\frac{(2m-2s+1)!!}{(2m-2s+4)!!}$$

To simplify (2.1), we need the following equality:

$$\sum_{m=1}^{n}(-1)^m m^s\binom{n}{m}=\begin{cases}-1, & s=0\\ 0, & s\geqslant 1,\quad n\geqslant s+1\end{cases} \tag{2.2}$$

which is known [5] for $s=0$ and $s=1$, and is easily proved by induction for $s\geqslant 2$. By using (2.2), the following equality [5] is proved:

$$\sum_{m=s}^{n-1}(-1)^m\binom{n}{m+1}\binom{m}{s}=(-1)^s \tag{2.3}$$

We reverse the order of summation for S_{1j}, S_{2j} in (2.1), and after simple manipulation, we apply (2.3) to the inner sums. We obtain the equalities

$$S_{1j}=\sum_{s=1}^{i-j}\frac{(2s-1)!!}{(2s+2)!!}u^{i-j-s},\qquad S_{2j}=\sum_{s=0}^{i-j}\frac{(2s+1)!!}{(2s+4)!!}u^{i-j-s} \tag{2.4}$$

Substituting (2.4) into (2.1), and again changing the order of summation in the remaining double sums, by using the equalities

$$\sum_{j=s}^{i}\frac{(-4)^j i}{i+j}\binom{i+j}{2j}a_{js}=(-1)^s4^{s-1}\binom{i+s-2}{2s-2}\frac{2i-1}{2s-1} \tag{2.5}$$

$$\sum_{j=s}^{i-1}(-1)^j4^{j+1}\binom{i+j}{2j}b_{js}=(-1)^s2^{2s-1}\binom{i+s-2}{2s-1}$$

$$\sum_{j=s}^{i-1}(-1)^j4^{j+1}\binom{i+j}{2j+1}a_{js}=(-1)^s2^{2s+1}\binom{i+s-1}{2s}$$

we write the final form of the simplified equality (2.1) after regrouping terms:

$$T_{2n}(\sqrt{u})=\frac{C_n}{K(k)}+\left[k'^2(u-1)+\frac{E(k)}{K(k)}\right]C_{n0}+\sum_{j=1}^{n-1}\Bigg[2^{2j-1}(2j+1)\times \tag{2.6}$$

$$k'^2u^{j+1}-\sum_{i=1}^{j}\frac{(-1)^{j-i}4^{i-1}(j+i-2)!}{(2i-1)!(j-i)!}p_j(k)u^i+(-1)^j2jk^2\Bigg]C_{nj}$$

$$p_j(k)=(4ij^2+i-1)(j-i+1)^{-1}k'^2-4j^2i^{-1}(j+i-1)k^2$$

Equating terms of identical power in u in (2.6), we obtain a system of $n+1$ equations in the coefficients C_n and C_{ni}

$$(2n-1)k'^2C_{nn-1}=4 \qquad (2.7)$$

$$(2i-1)k'^2C_{ni-1}+\frac{2(-1)^i}{(2i-1)!}\sum_{j=i}^{n-1}\frac{(-1)^j(j+i-2)!}{(j-i)!}p_j(k)\,C_{nj}=$$

$$(-1)^{n-i}\binom{n+i}{2i}\frac{8n}{n+i}\quad (i=n-1,\ n-2,\ldots,2)$$

$$2k'^2C_{n0}+8\sum_{j=1}^{n-1}(-1)^j j\,(k'^2-j^2k^2)\,C_{nj}=(-1)^{n-1}4n^2$$

$$\frac{8C_n}{K(k)}+8\left[\frac{E(k)}{K(k)}-k'^2\right]C_{n0}+16k^2\sum_{j=1}^{n-1}(-1)^j jC_{nj}=8(-1)^n$$

The coefficient matrix of the system (2.7) is a triangular one. The unknown C_{ni} are easily found one after the other, starting with C_{nn-1}. However, the system (2.7) can be simplified considerably. Multiplying both sides of (2.7) by

$$\binom{2i}{i-m}$$

we add the first $n-m+1$ equations. Then by using the identities

$$\sum_{i=m}^{j}\frac{d_{jm}}{(j-i)!}=\sum_{j=m}^{j-1}\frac{d_{jm}}{(j-i-1)!}=0,\quad d_{jm}=\frac{(-1)^i(j+i-3)!}{(i-m)!\,(i+m)!} \qquad (2.8)$$

$$(0\leqslant m\leqslant j-3)$$

$$\sum_{i=m}^{n}\frac{(-1)^i}{n+i}\binom{n+i}{2i}\binom{2i}{i-m}=0 \qquad (0\leqslant m\leqslant n-1)$$

(whose validity is proved by induction, as indeed was (2.5)), it can be shown that the coefficients of C_{ni} for $i\geqslant m+2$ and $0\leqslant m\leqslant n-3$, as well as the free terms for $0\leqslant m\leqslant n-1$ are zero. Adding the first two equations of the system (2.7) successively $(m=n-1)$, the three equations $(m=n-2)$ etc., we obtain the following recursion formulas for C_n and C_{ni} $(C_{ni}=0$ for $i\geqslant n)$:

$$(2n-1)k'^2C_{nn-1}=4,\qquad 4C_n+2q_{01}(C_{ni})=k'^2K(k)\,C_{n1} \qquad (2.9)$$

$$q_i(C_{ni})=-C_{n0}\ (i=1),\quad q_i(C_{ni})=0\qquad (i=2,3,\ldots,n-1)$$

$$q_{01}(C_{ni})=[2E(k)-k'^2K(k)]\,C_{n0}+k'^2K(k)\,C_{n1}$$

$$q_i(C_{ni})=(2i-1)k'^2C_{ni-1}+4i(1+k^2)\,C_{ni}+(2i+1)k'^2C_{ni+1}$$

Using (2.9), terms containing C_j and C_{ji} can generally be eliminated from equations of the system (1.12). Let us multiply the third equation of the system (1.12) by $k'^2K(k)$, the second by $2E(k)=k'^2K(k)$ and add them to the first equation multiplied by 4; then we multiply the fourth equation by $3k'^2$, the third one by $4(1+k^2)$ and we add them to the second equation multiplied by k'^2, etc. Consequently, taking account that $C_0=K(k)$, the system (1.12) takes the canonical form

$$a_i = \sum_{j=0}^{\infty} A_{ij}a_j + B_i \qquad (i = 0, 1, 2, \ldots) \tag{2.10}$$

$$A_{0j} = -q_{01}(e_{ij})[4K(k)]^{-1}, \qquad B_0 = [q_{01}(b_i) + 4\beta][4K(k)]^{-1}$$

$$A_{ij} = -{}^1/_4 q_i(e_{ij}), \quad B_i = {}^1/_4 q_i(b_i) \qquad (i = 1, 2 \ldots)$$

3. The approximate solution of the system (2.10) can be obtained by the method of reduction described in [3], say, which can also be given a foundation for the system mentioned. Let us limit ourselves to terms of degree not above $2n + 1$ in the integral equation for $\varphi_1(\xi)$ by expanding the function into a series. Then, the degree of the variable x in the left side of the equation will not be greater than $2n - 1$ because of (1.8). Hence, it is necessary to impose the constraint $i \leqslant n - 1$ on i in the expansion (1.9). The polynomials in the expansions of the functions are odd, hence, the condition for i, j in (1.9) can be written as $i + j \leqslant n - 1$, which is equivalent to the first constraint. Then if it is considered that $e_{ij} = 0$ for $i + j \geqslant n$, we write the reduced system of equations (2.10) as

$$a_i = \sum_{j=0}^{n-i} A_{ij}a_j + B_i \qquad (i = 0, 1, \ldots, n) \tag{3.1}$$

The coefficient matrix A_{ij} of the system (3.1) is almost triangular.

The integral characterisitics of the solution $\varphi_1(x)$ are of interest. The magnitude of the force P_1 is determined from the formulas

$$P_1 = \int_k^1 \varphi_1(x)\,dx = \sum_{i=0}^{n} a_i S_i, \quad S_i = \int_0^{\pi/2} \frac{\cos 2ix\,dx}{\sqrt{1 - k'^2 \sin^2 x}} \tag{3.2}$$

$$S_0 = K(k'), \quad S_1 = [2E(k') - (1 + k^2)K(k')](k')^{-2}$$

$$q_i(S_i) = 0 \qquad (i = 1, 2, \ldots)$$

The validity of the recursion formula in (3.2) is evident. We have a simple expression

$$M_1 = \int_k^1 \varphi_1(x)\,x\,dx = \frac{\pi}{2} a_0$$

for the moment M_1 because of (1.5), (1.6).

There remains to note that if the function $f(x)$ in (0.1) is representable by a rapidly converging series of the form (1.10) or its segment for $x \in [k, 1]$, then it is expedient to seek the solution $\varphi(x)$ not in the form (1.1) but using directly the scheme of the method of the polynomials $T_i^*(x)$.

As an illustration, let us examine the problem of the contact of two flat stamps ($f(x) = \operatorname{sgn} x$) with an elastic strip on a rigid base: hinge fixing holds on the boundaries of the stamp-strip base. This problem has no physical meaning, but is an important component part of the problems in which $f(x) = f_1(x) + \beta \operatorname{sgn} x$ and the function $f_1(x)$ is even.

The coefficients e_{ij} of the series (1.9) were computed on an electronic computer by successive approximations according to (1.11) by using the Gauss-Hermite quadrature formula and values of the function $F(t)$ tabulated in [6] which is a composite part of $G(\xi, x, \lambda)$. Because of the properties of the function $G(\xi, x, \lambda)$, the successive approximations must converge to the exact value of the integral. Values of $e_{ij} \cdot 10^4$ are presented in Table 1 for $\lambda = 2$ and $k = 0.1$, $k = 0.5$, $k = 0.9$.

The approximate solutions $\varphi(x)$ of the integral equation (0.1) are finally represented as

$$\varphi(x) = \frac{\operatorname{sgn} x}{g(x)} P_{2n}(x), \qquad P_{2n}(x) = \sum_{i=0}^{n} d_i x^{2i}$$

For the case $\lambda = 2$

$$P_{2n} = 0.636 + 0.205x^2 - 0.0650x^4 + 0.00792x^6, \qquad k = 0.1$$
$$P_{2n} = 0.569 + 0.194x^2 - 0.0592x^4 + 0.00512x^6, \qquad k = 0.5$$
$$P_{2n} = 0.383 + 0.136x^2 - 0.0310x^4, \qquad k = 0.9$$

Table 1

k	$ij = 00$	$ij = 10$	$ij = 11$	$ij = 20$	$ij = 21$	$ij = 22$	$ij = 30$
0.1	−2307	130.2	−19.17	−2.682	0.006966	0.006650	0.06120
0.5	−2244	93.53	−9.944	−1.409	0.04146	0.0010	0.01580
0.9	−2110	21.00	−0.5120	−0.06120			

Table 2

k	$\chi(k)$	$\varphi(1+k/2)$	$\chi(1)$	P	M
0.1	0.6414	1.533	0.7876	2.520	1.127
	0.6416	1.535	0.7784	2.514	1.123
0.5	0.7094	1.788	0.8210	1.418	1.046
	0.7098	1.786	0.8110	1.414	1.042
0.9	1.085	5.064	1.121	0.7954	0.7555
	1.083	5.038	1.111	0.7912	0.7514

Presented in Table 2 are some characterisitics of the solution $\varphi(x)$ for $\lambda = 2$, where the following notation has been introduced

$$\chi(k) = \lim \varphi(x) \sqrt{x^2 - k^2} \quad \text{for } x \to k$$
$$\chi(1) = \lim \varphi(x) \sqrt{1 - x^2} \quad \text{for } x \to 1$$

Values obtained by the "method of large λ" in [4] for the corresponding quantities are placed in the lower rows for comparison. The discrepancy between the values does not exceed 1.5% for all k.

As the parameter λ diminishes, especially for k close to zero, the number of equations in the system (3.1) must be increased to maintain given accuracy of the solution $\varphi(x)$. Computations have shown that the system method is effective for $\lambda > 1/4$ in

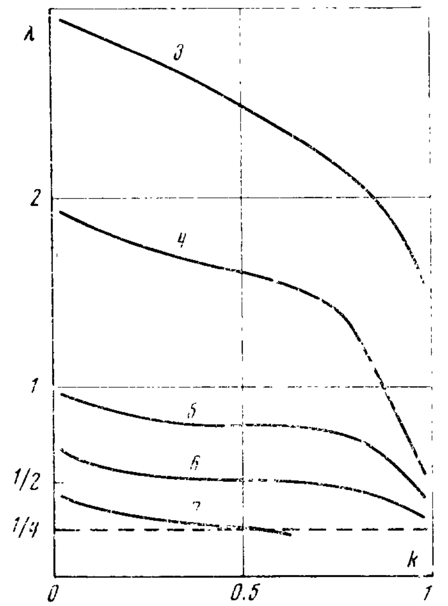

Fig. 1

this case for all $0 < k < 1$; the number of equations required in the system (3.1) hence does not exceed seven to obtain a solution $\varphi(x)$ to 2% accuracy.

The boundaries of applicability of the method of large λ ($\lambda \geqslant 2$), the system method ($\lambda > 1/4$) are shown in Fig. 1; domains in which the same number of equations of the system (3.1) is needed to obtain a solution $\varphi(x)$ to no less accuracy than 2% are separated by curved lines, where each curve is marked with the number of equations.

It should be noted that the solution of (0.1) found in closed form by using a special approximation of the kernel [4] differs from that obtained by the system method by not more than 5% for the case considered with $\lambda > 1/4$.

The author is grateful to V. M. Aleksandrov for monitoring the research and for useful discussions.

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Translated by M. D. F.